

The Energy of Scattering Solitons in the Ward Model

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Abstract

The energy density of a scattering soliton solution in Ward's integrable chiral model is shown to be instantaneously the same as the energy density of a static multi-lump solution of the \mathbb{CP}^3 sigma model. This explains the quantization of the total energy in the Ward model.

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1 Introduction

The Ward model (Ward 1988) is one of the most interesting models of soliton dynamics. It is defined in two space dimensions and is almost fully Lorentz invariant. The field equation can be regarded as the (2+1)-dimensional Bogomolny equation for Yang-Mills-Higgs fields, with gauge group $SU(2)$. This formulation is first order in time derivatives and is fully Lorentz invariant, but there is no natural Lagrangian and hence Noether's theorem is not available to construct conservation laws.

An alternative formulation, due to Ward, involves fixing the gauge, and this requires a choice of spatial direction. It leads to a field equation for an $SU(2)$ -valued “chiral” field J which is second order in time derivatives,

$$\partial_t(J^{-1}\partial_t J) - \partial_x(J^{-1}\partial_x J) - \partial_y(J^{-1}\partial_y J) + [J^{-1}\partial_y J, J^{-1}\partial_t J] = 0. \quad (1.1)$$

This equation is a variant of that of the standard chiral sigma model in 2+1 dimensions for J , where the final commutator term is absent. In this formulation, Lorentz invariance is broken to (1+1)-dimensional Lorentz invariance. However, there is a Lagrangian, and hence a conserved energy and a conserved momentum in the y -direction. One might imagine that by unfixing the gauge one could recover a complete conserved energy and momentum vector, but it is not clear how this should be done.

A remarkable feature of the Ward model is that it is integrable, both from the point of view of the (2+1)-dimensional Bogomolny equation and from the point of view of the equation for the chiral field J . The integrability ultimately arises from the fact that the Bogomolny equation is a reduction of the (2+2)-dimensional self-dual Yang-Mills equation, which can be treated by twistor methods. Because of the integrability there is a holomorphic structure, an infinite set of conservation laws (Ioannidou & Ward 1995), including the energy and momentum mentioned above, and a procedure for obtaining general solutions in terms of initial data using spectral methods and via the solution of a Riemann-Hilbert problem (Fokas & Ioannidou 2001). For us, the most useful consequence of integrability is that there are families of algebraically explicit solutions representing multi-soliton scattering processes.

These soliton solutions are constructed from auxiliary functions of a spectral parameter

λ , with poles. If there are a finite number of simple poles then the solution describes a finite number of solitons moving along straight lines in the plane at constant speed, and they do not scatter.

Ward (1995) noticed that if there is a double pole in λ then the solution represents scattering solitons. The double pole can be obtained (with care) in the limit that simple poles coalesce. The construction is technically a bit more difficult than the simple pole case. It is related to the construction of static solutions of the planar chiral sigma model, in terms of so-called unitons (Uhlenbeck 1989). Whereas the simple pole solutions correspond to single unitons, the double pole solutions correspond to 2-unitons. The original example of a double pole solution was generalised by Ioannidou (1996), and a number of interesting multi-soliton scattering solutions were found. The general solution of this type depends on an arbitrarily chosen pair of rational functions $f(z)$ and $h(z)$, where $z = x + iy$ is the complex coordinate in the spatial plane. Recently, a very general analysis of multi-pole, multi-soliton solutions has been given (Dai & Terng 2004).

In this paper we shall make some observations concerning the energy density and total energy of the general double pole solution discussed by Ward (1995) and Ioannidou (1996), which has no net momentum. Our proofs of these observations are not deep, and we expect that a better understanding is possible.

Our first observation is that the total energy of the double pole solutions is quantized in units of 8π . This was actually noted earlier by Ioannidou & Zakrzewski (1998) in a few examples, but no proof was given. It is physically quite a surprising result, because the solutions represent solitons in motion and part of the energy is kinetic. We are familiar with quantized energy for multi-solitons at rest, if they satisfy a Bogomolny equation, but usually any kinetic energy raises the total energy above the quantized value (see e.g. Manton & Sutcliffe 2004). We shall clarify below the number of energy units that a particular solution has – the relation to the number of solitons is not particularly simple.

Our second observation is that the energy density of the solutions (including both the gradient and kinetic energies of the field), calculated at any instant of time t , is the same as the energy density of a static $\mathbb{C}P^3$ sigma model solution (also known as a $\mathbb{C}P^3$ multi-lump).

The \mathbb{CP}^3 multi-lump has t as a parameter, but is not regarded as dynamical and has no kinetic energy. We have no deep understanding of this relationship between the $SU(2)$ Ward model solutions and the \mathbb{CP}^3 multi-lumps; we were led to it through the algebraic form of the energy density in a number of examples. Our proof that the energy density is the same is based on direct calculation.

The first observation follows from the second. It is well known that the total energy of a static \mathbb{CP}^3 multi-lump solution is quantized (Zakrzewski 1989), and we can determine how many units of energy there are in terms of the degrees of h , f , and the z -derivative of f .

2 Double Pole Solutions of the Ward Model

The simplest solutions of the Ward model arise from a function of λ with a single pole $\frac{R(z, \bar{z})}{\lambda - \mu}$, where μ is a constant, and $R(z, \bar{z})$ is a 2×2 matrix depending on a rational function $f(z)$. For $\mu = i$ this solution is static and is equivalent to the multi-lump solution of the \mathbb{CP}^1 sigma model given by the rational map $f(z)$. The topological charge N of the lump is the degree of f , and the energy in the Ward model is the same as in the sigma model, namely $8\pi N$.

The next simplest solutions in the Ward model are those based on a superposition of simple poles in the λ -plane. Let us denote the pole locations by $\mu_1, \mu_2, \dots, \mu_n$, which are necessarily distinct. Associated with these are rational functions $f_1(z), f_2(z), \dots, f_n(z)$. The function $f_r(z)$ describes a multi-lump which moves at a constant velocity determined by μ_r . Since the pole locations are distinct, so are the velocities, so at most one of the multi-lumps is at rest. Although the multi-lumps are in motion, they do not scatter off each other in this type of solution. There is a simple formula for the total energy, which varies continuously as the parameters μ_r vary, so the energy is not quantized.

Scattering soliton solutions are obtained by considering a limit of the solutions above, in which two or more of the pole locations μ_r are brought into coincidence. The simplest case is where $n = 2$ and the parameters μ_1 and μ_2 are brought into coincidence at i . If the limit is taken appropriately, a double pole at $\lambda = i$ results, and it is in this case that the momentum in the y -direction vanishes and the energy is quantized as a multiple of 8π . Solutions with a double pole at a different location are Lorentz boosted and have a different energy.

It was shown by Ward (1995) that for these double pole solutions the $SU(2)$ matrix J has a factorized 2-union form. We recall now the expression for J . It depends on two rational functions $f(z)$ and $h(z)$ (which arise from the functions $f_1(z)$ and $f_2(z)$ as the limit of coalescing poles is taken). Explicitly

$$J = \left(I - 2 \frac{q_1^\dagger \otimes q_1}{|q_1|^2} \right) \left(I - 2 \frac{q_2^\dagger \otimes q_2}{|q_2|^2} \right), \quad (2.1)$$

where I is the 2×2 unit matrix, and q_1 and q_2 are the 2-component row vectors

$$q_1 = (1 + |f|^2)(1, f) - 2i(tf' + h)(\bar{f}, -1), \quad (2.2)$$

$$q_2 = (1, f). \quad (2.3)$$

q_2 is just a function of z , but q_1 and hence J are functions of z , \bar{z} and t . f' denotes $\frac{df}{dz}$. Because of the time dependence of q_1 , the solution describes the scattering of solitons.

Ioannidou (1996) explored a number of examples of the 2-union solution (2.1). The simplest exhibiting soliton scattering is the solution with $f(z) = z$ and $h(z) = z^2$ (Ward 1995). The scattering can be deduced from the zeros of the expression $tf' + h = t + z^2$, noting that these are located at $z = \pm\sqrt{-t}$. The zeros are on the x -axis for $t < 0$, pass through the origin at $t = 0$, and are on the y -axis for $t > 0$. A slightly more complicated example is with $f(z) = z$ and $h(z) = z^3$. Here three solitons scatter with equilateral triangular symmetry. Solutions with $h = 0$ are also possible. If $f(z) = z$ and $h = 0$, for example, then there is circular symmetry for all t , with a ring-like soliton contracting and then expanding.

It is rather curious that these solutions (and also their energy density, which we investigate in the next section) appear to respect the rotational symmetry of the plane, despite our earlier remarks about how Lorentz invariance is partially broken. It is possible that these solutions have zero momentum and the same energy no matter which direction is chosen to fix the gauge of the Bogomolny equation, but we have not established this.

3 Energy of 2-Union Solutions

The energy density of the field J in the Ward model is

$$\mathcal{E}_J = -\frac{1}{2} \text{Tr}((J^{-1} \partial_t J)^2 + (J^{-1} \partial_x J)^2 + (J^{-1} \partial_y J)^2). \quad (3.1)$$

This is actually the same as in the standard Lorentz invariant sigma model – it is the field equation of the Ward model that breaks the Lorentz invariance. If we use the complex coordinate $z = x + iy$ then

$$\mathcal{E}_J = -\frac{1}{2}\text{Tr}((J^{-1}\partial_t J)^2 + 4(J^{-1}\partial_z J)(J^{-1}\partial_{\bar{z}} J)). \quad (3.2)$$

The contribution $-\frac{1}{2}\text{Tr}(J^{-1}\partial_t J)^2$ is the kinetic energy density; the remaining gradient terms can be thought of as the potential energy density. The total energy is $E = \int \mathcal{E}_J d^2x$.

Our main result is that this energy density is the same as that of a \mathbb{CP}^3 sigma model lump solution. In the \mathbb{CP}^3 sigma model (Zakrzewski 1989), the basic field is a 4-vector of complex functions in the plane

$$V = (V_1, V_2, V_3, V_4), \quad (3.3)$$

whose energy density and other properties are unaffected by multiplying all components by a common function W . The components of V should not simultaneously vanish. For a finite-energy multi-lump solution, all components of V must be rational functions of z . V can then be converted to a 4-vector of polynomials in z , with no common root, by multiplying through by the common denominator. (For an anti-lump, one takes rational functions of \bar{z} instead. There are also higher energy saddle point solutions which depend on z and \bar{z} , but these do not concern us here.)

The energy density in the sigma model is

$$\mathcal{E}_V = 8 \frac{\sum_{1 \leq i < j \leq 4} |V_i V'_j - V_j V'_i|^2}{(|V_1|^2 + |V_2|^2 + |V_3|^2 + |V_4|^2)^2}, \quad (3.4)$$

where $V'_i = \frac{dV_i}{dz}$. The topological charge N of the lump is the degree of V , $\deg V$, which is the highest degree among the polynomials comprising V (after clearing denominators), and the total energy $\int \mathcal{E}_V d^2x$ is $8\pi N$. This follows from a Bogomolny argument which reduces the energy to 8π times the integral over the plane of the pull-back of the 2-form generating the integer cohomology ring of the manifold \mathbb{CP}^3 .

The multi-lump that corresponds to the 2-unton solution of the Ward model has the specific form

$$V = (2(tf' + h), f^2, \sqrt{2}f, 1). \quad (3.5)$$

Note that t appears here explicitly as a parameter, but we calculate the energy density at each instant as if the multi-lump were static, using the formula (3.4).

To find the total energy, we need to determine the highest degree of the polynomials that occur in (3.5) when one clears denominators. Let $\deg f$ and $\deg h$ denote the degrees of the rational functions f and h . Generically, $f = \frac{r}{s}$ and $h = \frac{u}{v}$, where r and s are polynomials of degree $\deg f$, u and v are polynomials of degree $\deg h$, and s and v have no common roots. Then substituting in (3.5) and multiplying by the common denominator s^2v , we find

$$V = (2t(sr'v - rs'v) + 2us^2, r^2v, \sqrt{2}rsv, s^2v), \quad (3.6)$$

so $N = 2\deg f + \deg h$. However there are plenty of non-generic possibilities. For example, if f and h are polynomials, then N is the greater of $2\deg f$ and $\deg h$. Generically, N is also the number of zeros of $tf' + h$, but again there are non-generic examples where this is not the case.

Now consider the energy density for V of the form (3.5). Since time derivatives play no role here, let us denote $2(tf' + h)$ by g . One finds after some algebra that

$$\mathcal{E}_V = \frac{8|(1 + |f|^2)g' - 2g\bar{f}f'|^2 + 16|gf'|^2 + 16(1 + |f|^2)^2|f'|^2}{(|g|^2 + (1 + |f|^2)^2)^2}. \quad (3.7)$$

We now wish to demonstrate that $\mathcal{E}_J = \mathcal{E}_V$. Let us consider the kinetic contribution to \mathcal{E}_J first. Note that $J = AB$ (the product of the two uniton factors in (2.1)), and that $A^{-1} = A$ and $B^{-1} = B$. Note also that B is independent of t . Hence $\text{Tr}(J^{-1}\partial_t J)^2 = -\text{Tr}(\partial_t A \partial_t A)$. Further simplification is possible by expressing q_1 as a linear combination of the everywhere orthonormal, time independent row vectors $(1 + |f|^2)^{-\frac{1}{2}}(1, f)$ and $(1 + |f|^2)^{-\frac{1}{2}}(\bar{f}, -1)$. Their coefficients are, respectively, $(1 + |f|^2)^{\frac{1}{2}}$ and $-2i(1 + |f|^2)^{-\frac{1}{2}}g$. After a modest calculation involving these coefficients, and noting that the time derivative of g is $2f'$, one finds that

$$-\frac{1}{2}\text{Tr}(J^{-1}\partial_t J)^2 = \frac{16(1 + |f|^2)^2|f'|^2}{(|g|^2 + (1 + |f|^2)^2)^2}, \quad (3.8)$$

which is identical to the contribution to \mathcal{E}_V from the third term in the numerator of (3.7). A rather more involved calculation, using MAPLE, leads to the result that the gradient energy is

$$-2\text{Tr}(J^{-1}\partial_z J)(J^{-1}\partial_{\bar{z}} J) = \frac{8|(1 + |f|^2)g' - 2g\bar{f}f'|^2 + 16|gf'|^2}{(|g|^2 + (1 + |f|^2)^2)^2}, \quad (3.9)$$

which is identical to the contribution of the first and second terms in the numerator of (3.7). In total, we see that $\mathcal{E}_J = \mathcal{E}_V$, and note that in \mathcal{E}_J we include time derivatives, but in \mathcal{E}_V we do not.

Let us illustrate these formulae with a pair of examples. The first is the two-soliton solution, with $f(z) = z$ and $h(z) = z^2$. The energy density is

$$\mathcal{E} = 16 \frac{5r^4 + 10r^2 + 1 + 4t^2(2r^2 + 1) - 8t(x^2 - y^2)}{(5r^4 + 2r^2 + 1 + 4t^2 + 8t(x^2 - y^2))^2}, \quad (3.10)$$

where $r^2 = x^2 + y^2$. The energy is peaked at two points on the x -axis for $t < 0$, forms a ring around the origin at $t = 0$, and is peaked at two points on the y -axis for $t > 0$. See Ward (1995) for figures showing the energy distribution at different times. The total energy is 16π , because the highest degree of the polynomials in V is 2. This is also easily verified by direct integration of the circularly symmetric energy density at $t = 0$, and at other times follows from energy conservation.

A second example is $f(z) = z$ and $h(z) = z^3$. Here the energy density is

$$\mathcal{E} = 16 \frac{2r^8 + 16r^6 + 19r^4 + 2r^2 + 1 + 4t^2(2r^2 + 1) - 8t(r^2 + 2)x(x^2 - 3y^2)}{(4r^6 + r^4 + 2r^2 + 1 + 4t^2 + 8tx(x^2 - 3y^2))^2}, \quad (3.11)$$

(which simplifies and corrects formula (21) of Ioannidou (1996)). This energy density exhibits the scattering of three lumps with triangular symmetry, whose locations are approximately at the zeros of $z^3 + t$ (see Ioannidou (1996) for figures). The total energy is 24π .

We have also investigated other cases where f and h are rational functions, but not polynomials. Figure 1 presents an example of generic type, where $f = 1/z$ and $h = 1/(z - 1)$. The expression for the energy density is complicated, but it is easily seen that $\deg V = 3$, implying that the total energy is 24π (as confirmed numerically). In addition, the motion of the configuration is interesting as seen from the zeroes of $tf' + h$, that is, the zeroes of $z^2 - tz + t$. These show the locations of a pair of soliton structures whose motion is not so simple. For $t < 0$ and $t > 4$ both zeros are real, but for $0 < t < 4$ the zeros are a complex conjugate pair. At $t = 0$ and $t = 4$ there are repeated zeros at $z = 0$ and $z = 2$, respectively. In Figure 1 we can see solitons approaching along the x -axis at early times, colliding at $t = 0$ and scattering at right angles, then moving apart on circular orbits and colliding again at $t = 4$, and finally separating along the x -axis at late times.

An example of the non-generic type (because the denominator of h is the square of the denominator of f) is $f(z) = 1/z$ and $h(z) = (z - 1)/z^2$. Here, the energy density is

$$\mathcal{E} = 16 \frac{3r^4 + 2r^2 + 3 + 8\tau^2 r^2 + 4\tau^2 - 8\tau r^2 x}{(r^4 + 6r^2 + 1 + 4\tau^2 - 8\tau x)^2}, \quad (3.12)$$

where $\tau = t + 1$. This is symmetric under the simultaneous reflections $x \rightarrow -x$ and $\tau \rightarrow -\tau$. Figure 2 presents the evolution of the configuration for times around $t = -1$. The total energy is 16π .

4 Generalisations

More complicated multi-pole solutions of the Ward model can be constructed. For a recent, general analysis of these, see Dai & Terng (2004). A triple-pole example, for which J has a 3-union form, is presented in Ioannidou (1996). In this, and a few further examples, we have verified that the total energy is an integer multiple of 8π . We suspect that the instantaneous energy density can again be identified with that of a static sigma model solution, but now for $\mathbb{C}P^5$. Clarification of this, and its generalisation, would be desirable.

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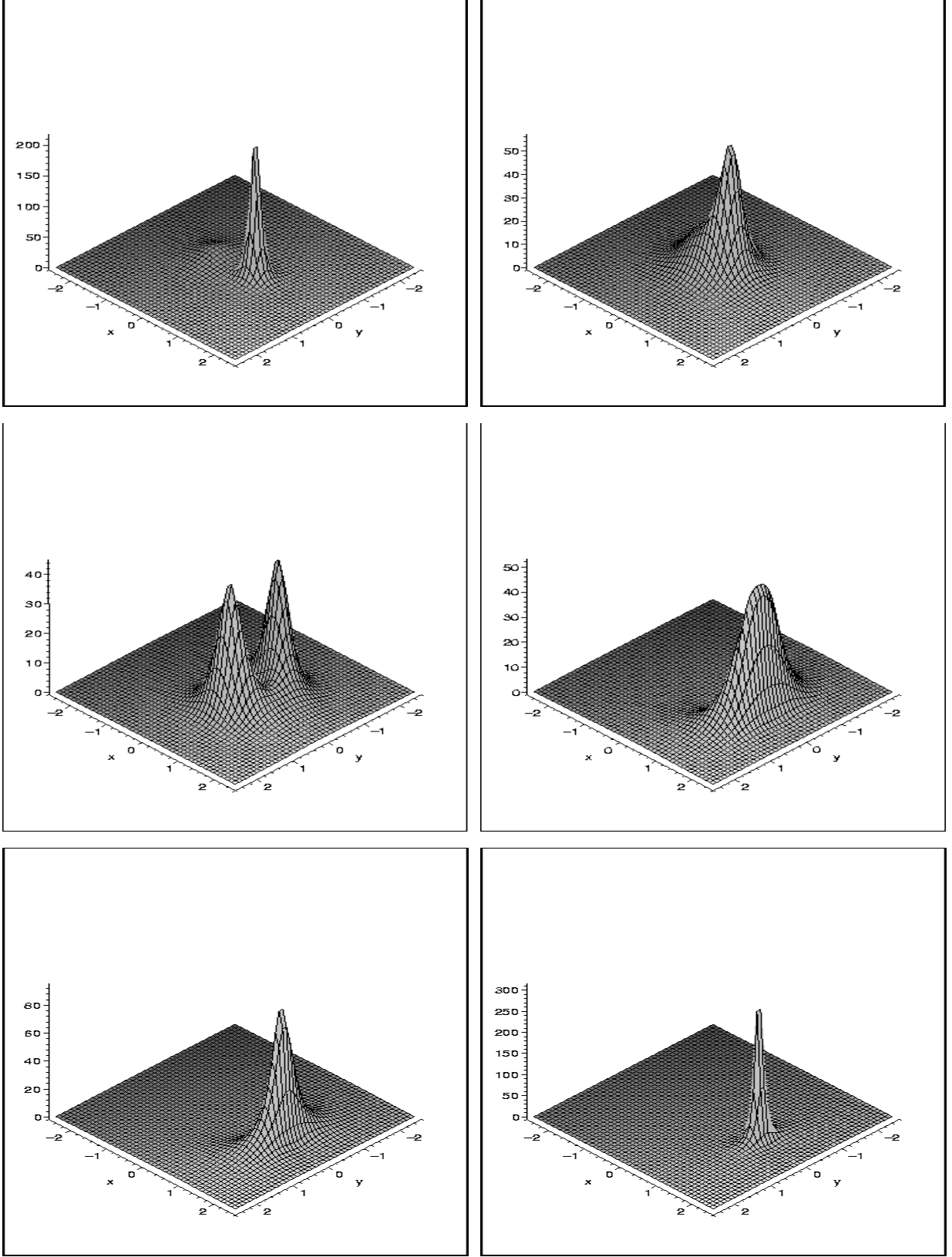


Figure 1: The soliton configuration at different times for the generic example $f = 1/z$ and $h = 1/(z - 1)$.

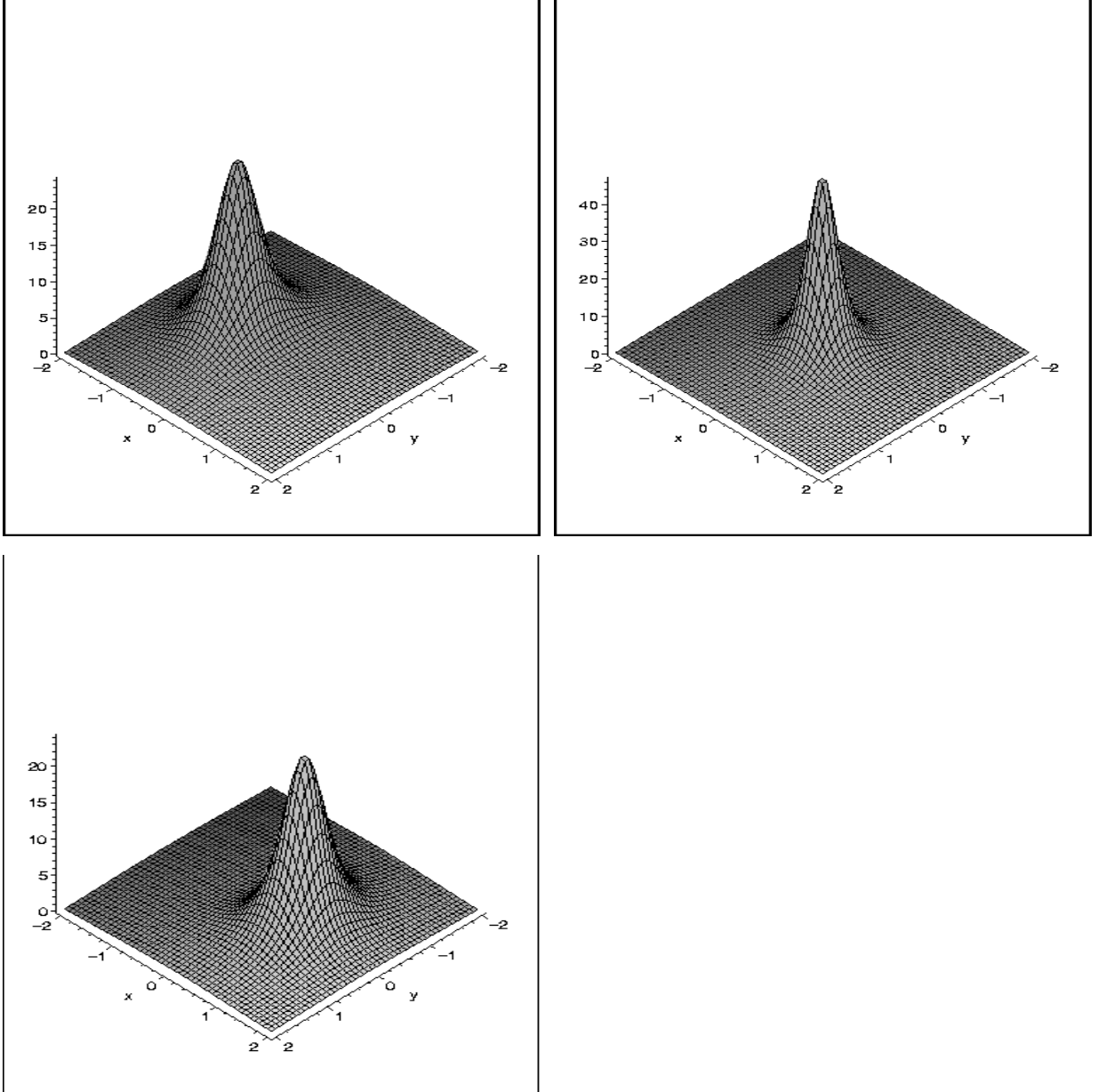


Figure 2: The soliton configuration at different times for the non-generic example $f = 1/z$ and $h = (z - 1)/z^2$.